

The singular locus of Lauricella's F_C

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Abstract. We determine the singular locus of the Lauricella function F_C by utilizing the theory of D -modules and Gröbner basis. The A -hypergeometric system associated to F_C is also discussed.

1 Introduction

Let $D = \mathbb{C}\langle x_1, \dots, x_m, \partial_1, \dots, \partial_m \rangle$ be the Weyl algebra of m variables. We take $2m$ dimensional integer vector (u, v) , $u, v \in \mathbb{Z}^m$ such that $u_i + v_i > 0$. For an element $p = \sum_{(\alpha, \beta) \in E} c_{\alpha, \beta} x^\alpha \partial^\beta$ of D , we define the (u, v) -initial form $\text{in}_{(u, v)}(p)$ of p by the sum of the terms in p which has the highest (u, v) -weight. In other words, we define

$$\begin{aligned} \text{ord}_{(u, v)}(p) &= \max_{(\alpha, \beta) \in E} (\alpha \cdot u + \beta \cdot v), \\ \text{in}_{(u, v)}(p) &= \sum_{(\alpha, \beta) \in E, \alpha \cdot u + \beta \cdot v = \text{ord}_{(u, v)}(p)} c_{\alpha, \beta} x^\alpha \xi^\beta. \end{aligned}$$

Here, ξ_i is a new variable which commutes with the other variables (see, e.g., [12, §1.1]). When $u_i + v_i = 0$, we define the (u, v) -initial form analogously and ξ_i is replaced by ∂_i in the definition above. Put $\mathbf{0} = (0, \dots, 0) \in \mathbb{Z}^m$ and $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}^m$. For $p \in D$, the $(\mathbf{0}, \mathbf{1})$ -initial form of p is called the principal symbol of p . For a given left ideal I of D , the characteristic ideal of I is $\text{in}_{(\mathbf{0}, \mathbf{1})}(I)$, which is the ideal in $\mathbb{C}[x_1, \dots, x_m, \xi_1, \dots, \xi_m]$ generated by all principal symbols of the elements of I .

The zero set in \mathbb{C}^{2m} of the characteristic ideal is called the characteristic variety, which is denoted by $\text{Ch}(I)$. When the (Krull) dimension of the characteristic variety is m , then D/I is called a holonomic D -module and the I is called a holonomic ideal. The projection of $\text{Ch}(I) \setminus V(\xi_1, \dots, \xi_m)$ to the first m -coordinates $\mathbb{C}^m = \{x\}$ is called the singular locus of I , which is denoted by $\text{Sing}(I)$ or $\text{Sing}(D/I)$. As to these fundamental notions of D -modules, see, e.g., [2].

Let $R = \mathbb{C}(x_1, \dots, x_m)\langle \partial_1, \dots, \partial_m \rangle$ be the ring of differential operators with rational function coefficients. The holonomic rank of I is the dimension of $R/(RI)$ regarded as the $\mathbb{C}(x) = \mathbb{C}(x_1, \dots, x_m)$ vector space. The rank is denoted

by $\text{rank}(I)$. The rank is equal to the multiplicity of the characteristic ideal at a generic point. In other words, we have

$$\text{rank}(I) = \dim_{\mathbb{C}} \mathcal{O}_a / \mathcal{O}_a \cdot \text{in}_{(\mathbf{0}, \mathbf{1})}(I)|_{x=a},$$

where $\mathcal{O}_a = \mathbb{C}\{\xi_1 - a_1, \dots, \xi_m - a_m\}$ and a is a point in $\mathbb{C}^m \setminus \text{Sing}(I)$. We also have the identity

$$\text{rank}(I) = \dim_{\mathbb{C}(x)} \mathbb{C}(x)[\xi] / \mathbb{C}(x)[\xi] \cdot \text{in}_{(\mathbf{0}, \mathbf{1})}(I)$$

where $\mathbb{C}(x)[\xi]$ denotes $\mathbb{C}(x_1, \dots, x_m)[\xi_1, \dots, \xi_m]$. Let $\text{Sol}(I)$ be the constructive sheaf of holomorphic solutions on \mathbb{C}^m ;

$$\text{Sol}(I) = \{f \in \mathcal{O} \mid \ell \cdot f = 0 \text{ for all } \ell \in I\}.$$

The holonomic rank $\text{rank}(I)$ is equal to $\dim_{\mathbb{C}} \text{Sol}(I)(U)$ for any simply connected open set U in $\mathbb{C}^m \setminus \text{Sing}(I)$. As to these characterizations of the holonomic rank, see, e.g., [9], [12, Chapter 1] and their references.

Put $\theta_i = x_i \partial_i$ and $\theta = \sum_{i=1}^m \theta_i$. We consider the left ideal $I(m)$ generated by the operators

$$\ell_i = \theta_i(\theta_i + c_i - 1) - x_i(\theta + a)(\theta + b), \quad i = 1, \dots, m, \quad (1)$$

where $a, b, c_i \in \mathbb{C}$ are parameters. The Lauricella function F_C is annihilated by the left ideal.

We will show, in Theorem 4.3, that the singular locus of $I(m)$ agrees with the zero set of

$$\prod_{i=1}^m x_i \prod_{\varepsilon_i \in \{-1, 1\}} (1 + \varepsilon_1 \sqrt{x_1} + \dots + \varepsilon_m \sqrt{x_m}). \quad (2)$$

Note that when we expand (2), then it becomes a polynomial in x . The proof of this fact occupies the first three sections of this paper. In the last section, we study the A -hypergeometric system associated to the Lauricella F_C and determine the singular locus of it in the complex torus by utilizing our main theorem 4.3.

We had believed that the singular locus of $I(m)$ is well-known among experts, but we find a few literatures on rigorous proofs on these facts. Some of them are master theses by Kaneko [7] and by Yoshida [15], who proved that the singular locus is contained in (2) but they did not discuss on the opposite inclusion.

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2 A variety containing the singular locus

The singular locus of the system $I(m)$ is, by definition, $\pi(V(\text{in}_{(\mathbf{0}, \mathbf{1})}(I(m)) \setminus V(\xi_1, \dots, \xi_m)))$. Here, π is the projection from (x, ξ) to x . The principal symbol of ℓ_i is denoted

by L_i and they are equal to

$$L_i = x_i^2 \xi_i^2 - x_i \left(\sum_{j=1}^m x_j \xi_j \right)^2.$$

In other words, $\text{in}_{(0,1)}(\ell_i) = L_i$. Since $L_i \in \text{in}_{(0,1)}(I(m))$, the singular locus of the system $I(m)$ is contained in $C = \pi(V(L_1, \dots, L_m) \setminus V(\xi_1, \dots, \xi_m))$.

Let us regard $V(L_1, \dots, L_m)$ as an analytic space. When $x_i \neq 0$, L_i is factored as

$$L_i = \left(x_i \xi_i - \sqrt{x_i} \left(\sum_{j=1}^m x_j \xi_j \right) \right) \left(x_i \xi_i + \sqrt{x_i} \left(\sum_{j=1}^m x_j \xi_j \right) \right). \quad (3)$$

Therefore, the necessary and sufficient condition that x lies in $C \cap (\mathbb{C}^*)^m$ is that

$$x_i \xi_i + \varepsilon_i \sqrt{x_i} \left(\sum_{j=1}^m x_j \xi_j \right) = 0, \quad i = 1, \dots, m \quad (4)$$

has a non-trivial solution $\xi \neq 0$ for a set of signs ε_i . The condition can be written in terms of the determinant of the system regarded as a system of linear equations with respect to ξ .

Proposition 2.1 *The determinant of the coefficient matrix of the system of linear equation (4) is equal to $\prod_{i=1}^m x_i \left(1 + \sum_{j=1}^m \varepsilon_j \sqrt{x_j} \right)$.*

Proof. The coefficient matrix of the system (4) is

$$M = \begin{pmatrix} x_1 + \varepsilon_1 x_1 \sqrt{x_1} & \varepsilon_1 x_2 \sqrt{x_1} & \varepsilon_1 x_3 \sqrt{x_1} & \cdots & \varepsilon_1 x_m \sqrt{x_1} \\ \varepsilon_2 x_1 \sqrt{x_2} & x_2 + \varepsilon_2 x_2 \sqrt{x_2} & \varepsilon_2 x_3 \sqrt{x_2} & \cdots & \varepsilon_2 x_m \sqrt{x_2} \\ \varepsilon_3 x_1 \sqrt{x_3} & \varepsilon_3 x_2 \sqrt{x_3} & x_3 + \varepsilon_3 x_3 \sqrt{x_3} & \cdots & \varepsilon_3 x_m \sqrt{x_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon_m x_1 \sqrt{x_m} & \varepsilon_m x_2 \sqrt{x_m} & \varepsilon_m x_3 \sqrt{x_m} & \cdots & x_m + \varepsilon_m x_m \sqrt{x_m} \end{pmatrix}.$$

We have

$$\begin{aligned}
& \det M \\
&= x_1 x_2 \cdots x_n \det \begin{pmatrix} 1 + \varepsilon_1 \sqrt{x_1} & \varepsilon_1 \sqrt{x_1} & \varepsilon_1 \sqrt{x_1} & \cdots & \varepsilon_1 \sqrt{x_1} \\ \varepsilon_2 \sqrt{x_2} & 1 + \varepsilon_2 \sqrt{x_2} & \varepsilon_2 \sqrt{x_2} & \cdots & \varepsilon_2 \sqrt{x_2} \\ \varepsilon_3 \sqrt{x_3} & \varepsilon_3 \sqrt{x_3} & 1 + \varepsilon_3 \sqrt{x_3} & \cdots & \varepsilon_3 \sqrt{x_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon_m \sqrt{x_m} & \varepsilon_m \sqrt{x_m} & \varepsilon_m \sqrt{x_m} & \cdots & 1 + \varepsilon_m \sqrt{x_m} \end{pmatrix} \\
&= \prod_{i=1}^m x_i \det \begin{pmatrix} 1 + \varepsilon_1 \sqrt{x_1} & -1 & -1 & \cdots & -1 \\ \varepsilon_2 \sqrt{x_2} & 1 & 0 & \cdots & 0 \\ \varepsilon_3 \sqrt{x_3} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon_m \sqrt{x_m} & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} [j\text{th column}] \\ +[1\text{st column}] \times (-1) \\ \text{for } 2 \leq j \leq m \end{pmatrix} \\
&= \prod_{i=1}^m x_i \det \begin{pmatrix} 1 + \sum_{j=1}^m \varepsilon_j \sqrt{x_j} & 0 \\ * & E_{m-1} \end{pmatrix} \begin{pmatrix} [1\text{st row}] + \sum_{j=2}^m [j\text{th row}] \end{pmatrix} \\
&= \prod_{i=1}^m x_i \left(1 + \sum_{j=1}^m \varepsilon_j \sqrt{x_j} \right)
\end{aligned}$$

Q.E.D.

Theorem 2.2 [7], [15] *The singular locus of $I(m)$ is contained in the zero set of (2).*

Proof. Since $x_i = 0$ are already the zero set of (2), we may only consider the singular locus in $(\mathbb{C}^*)^m$. If $x \in (\mathbb{C}^*)^m \cap C$, then (4) must have a non-trivial solution $\xi \neq 0$. By Proposition 2.1, we get the conclusion. Q.E.D.

In the sequel, we want to prove the opposite inclusion $C \subseteq \text{Sing}(I)$ where C is the zero set of (2). If a classical solution of $I(m)$ has singularities on the all irreducible components of the zero set C , then we are done. However, as the following example shows, the singular locus of classical solutions may smaller than the zero set C .

Example 2.3 Assume $m = 2$. When $a = -1/2, b = -2, c_1 = c_2 = 1/2$, the solution space of the differential equation is spanned by the following functions

$$1 + 2x + 2y - 2xy - x^2/3 - y^2/3, \sqrt{x}, \sqrt{y}, \sqrt{xy}(1 - x/3 - y/3)$$

Note that the singular locus of these solutions is contained in $xy = 0$, which is smaller than the zero set C .

We close this section with two preparatory propositions.

Proposition 2.4 *The left ideal $I(m)$ is holonomic.*

Proof. Since the Bernstein inequality $\dim V(\text{in}_{(0,1)}(I(m))) \geq m$ holds (see, e.g., [2], [12]), we may prove that $\dim V(\text{in}_{(0,1)}(I(m))) \leq m$. It follows from the decomposition (3) that for any $x \in (\mathbb{C}^*)^m$ the ξ 's satisfy (3) are finite points. Hence, the analytic set $C \cap (\mathbb{C}^*)^m$ is m -dimensional. The remaining thing to do is the evaluation of the dimension at the points in $x_i = 0$.

We put $I'(m) = \langle L_1, \dots, L_m \rangle$. $I'(m)$ is contained in $\text{in}_{(0,1)}(I(m))$. We will prove $\dim I'(m) = m$ by induction. When $m = 1$, it is easy to see that $\dim I'(m) = 1$. Let us assume $\dim I'(m-1) = m-1$. We note that $V(I'(m)) \cap V(x_m) = \{((x', \xi'), (0, \xi_m)) \mid (x', \xi') \in V(I'(m-1)) \subset \mathbb{C}^{m-1}, \xi_m \in \mathbb{C}\}$ because $x_m \xi_m = 0$ in L_i when $x_m = 0$. It follows from the induction hypothesis $\dim V(I'(m-1)) = m-1$ that the dimension of $V(I'(m))$ at any point in $x_m = 0$ is $(m-1) + 1 = m$. Q.E.D.

Proposition 2.5 *The polynomial*

$$\prod_{\varepsilon_i \in \{+1, -1\}} (1 + \varepsilon_1 \sqrt{x_1} + \dots + \varepsilon_m \sqrt{x_m})$$

is irreducible in $\mathbb{C}[x_1, \dots, x_m]$.

Proof. Let

$$\begin{aligned} P(t) &= P(t_1, \dots, t_m) = \prod_{\varepsilon \in \{\pm 1\}^m} P_\varepsilon(t), \\ P_\varepsilon(t) &= 1 + \varepsilon_1 t_1 + \varepsilon_2 t_2 + \dots + \varepsilon_m t_m, \end{aligned}$$

where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$. First we show that the polynomial $P(t)$ is irreducible in the subring $\mathbb{C}[t_1^2, t_2^2, \dots, t_m^2]$ of $\mathbb{C}[t]$. Suppose that there exist $Q_1(t), Q_2(t) \in \mathbb{C}[t]$ satisfying

$$P(t) = Q_1(t_1^2, t_2^2, \dots, t_m^2) Q_2(t_1^2, t_2^2, \dots, t_m^2).$$

Then in $\mathbb{C}[t]$, we have in particular

$$P_1(t) \mid Q_1(t_1^2, t_2^2, \dots, t_m^2) Q_2(t_1^2, t_2^2, \dots, t_m^2),$$

where $\mathbf{1} = (1, 1, \dots, 1) \in \{\pm 1\}^m$. Since $P_1(t) = 1 + t_1 + t_2 + \dots + t_m$ is a prime element in $\mathbb{C}[t]$, either $Q_1(t_1^2, t_2^2, \dots, t_m^2)$ or $Q_2(t_1^2, t_2^2, \dots, t_m^2)$ is divided by $P_1(t)$ in $\mathbb{C}[t]$. Assume the former, then we have

$$Q_1(t_1^2, t_2^2, \dots, t_m^2) = P_1(t) g(t) \tag{5}$$

for some $g(t) \in \mathbb{C}[t]$. For any $\varepsilon \in \{\pm 1\}^m$, substitute $t = (\varepsilon_1 t_1, \dots, \varepsilon_m t_m)$ in the equation (5). Then we obtain

$$Q_1(t_1^2, t_2^2, \dots, t_m^2) = P_\varepsilon(t) g(\varepsilon_1 t_1, \dots, \varepsilon_m t_m).$$

Thus the polynomial $Q_1(t_1^2, t_2^2, \dots, t_m^2)$ is divided by $P_\varepsilon(t)$ for any $\varepsilon \in \{\pm 1\}^m$, which implies $P(t) \mid Q_1(t_1^2, t_2^2, \dots, t_m^2)$. Therefore we conclude that $P(t)$ is irreducible in $\mathbb{C}[t_1^2, t_2^2, \dots, t_m^2]$.

Next we prove that the polynomial

$$\prod_{\varepsilon_i \in \{+1, -1\}} (1 + \varepsilon_1 \sqrt{x_1} + \cdots + \varepsilon_m \sqrt{x_m}) = \prod_{\varepsilon \in \{\pm 1\}^m} P_\varepsilon(\sqrt{x_1}, \dots, \sqrt{x_m}) \in \mathbb{C}[x]$$

is irreducible in $\mathbb{C}[x]$. Suppose that

$$\prod_{\varepsilon \in \{\pm 1\}^m} P_\varepsilon(\sqrt{x_1}, \dots, \sqrt{x_m}) = q(x)r(x) \quad (q(x), r(x) \in \mathbb{C}[x]).$$

Substitute $x = (t_1^2, t_2^2, \dots, t_m^2)$ in the above, then we have the equation

$$P(t) = q(t_1^2, t_2^2, \dots, t_m^2)r(t_1^2, t_2^2, \dots, t_m^2).$$

in $\mathbb{C}[t]$. Hence we may assume that $q(t_1^2, t_2^2, \dots, t_m^2)$ is a constant, so is $q(x) \in \mathbb{C}[x]$. Therefore the proof is completed. Q.E.D.

3 The singular locus in the complex torus

If we can show that the set $\{L_1, \dots, L_m\}$ is a set of generators of the characteristic ideal $\text{in}_{(\mathbf{0}, \mathbf{1})}(I(m))$, then we conclude that the singular locus agrees with the zero set of (2). However, it seems not to be easy to prove it. Instead of trying to prove it, we will firstly determine the characteristic variety in the complex torus.

We consider the left ideal $I'(m)$ generated by

$$\ell'_i = y_i \theta_i (\theta_i - c_i + 1) - (\theta - a)(\theta - b), \quad i = 1, \dots, m.$$

Here, $\theta_i = y_i \partial_{y_i}$ and $\theta = \theta_1 + \cdots + \theta_m$. These operators are obtained by applying the change of the coordinates $y_i = 1/x_i$, $i = 1, \dots, n$ to ℓ_i 's and multiplying y_i to them. The ring of differential operators with respect to the variable y is also denoted by D as long as no confusion arises. The characteristic varieties of $I(m)$ and $I'(m)$ agree on the complex torus $(\mathbb{C}^*)^m$ under the change of coordinates $y_i = 1/x_i$.

We use the order \succ_w defined by the first weight vector $w^{(1)} = (\mathbf{0}, \mathbf{1})$ and the second weight vector $w^{(2)} = (\mathbf{1}, \mathbf{0})$. In other words, $y^\alpha \partial^\beta \succ_w y^{\alpha'} \partial^{\beta'}$ if and only if

$$\begin{aligned} & (\alpha, \beta) \cdot w^{(1)} > (\alpha', \beta') \cdot w^{(1)} \\ \text{or } & \left((\alpha, \beta) \cdot w^{(1)} = (\alpha', \beta') \cdot w^{(1)} \text{ and } (\alpha, \beta) \cdot w^{(2)} > (\alpha', \beta') \cdot w^{(2)} \right) \\ \text{or } & \left((\alpha, \beta) \cdot w^{(j)} = (\alpha', \beta') \cdot w^{(j)} \text{ } (j = 1, 2) \text{ and } (\alpha, \beta) >_{\text{lex}} (\alpha', \beta') \right). \end{aligned}$$

We denote by $\text{in}_{\prec_w}(f)$ the leading monomial of $f \in D$ with respect to the order \prec_w . For two elements $f, g \in D$ with

$$\text{in}_{\prec_w} f = f_{\alpha\beta} y^\alpha \xi^\beta, \quad \text{in}_{\prec_w} g = g_{\alpha'\beta'} y^{\alpha'} \xi^{\beta'},$$

we define the S -pair $\text{sp}(f, g)$ of f and g by

$$\text{sp}(f, g) = g_{\alpha'\beta'} y^{\gamma-\alpha} \partial^{\delta-\beta} f - f_{\alpha\beta} y^{\gamma-\alpha'} \partial^{\delta-\beta'} g,$$

where

$$\gamma = (\max\{\alpha_1, \alpha'_1\}, \dots, \max\{\alpha_m, \alpha'_m\}), \quad \delta = (\max\{\beta_1, \beta'_1\}, \dots, \max\{\beta_m, \beta'_m\}).$$

For a subset G of D , the relation $f = \sum c_i g_i, g_i \in G$ is called a standard representation of f with respect to G when $c_i g_i \preceq_w f$ holds for all i such that $c_i \neq 0$.

Proposition 3.1 *The characteristic ideal $\text{in}_{(\mathbf{0}, \mathbf{1})}(I'(m))$ is generated by $\text{in}_{(\mathbf{0}, \mathbf{1})}(\ell'_i)$, $i = 1, \dots, m$.*

Proof. We use the order \succ_w . Since $\text{in}_{\prec_w}(\ell'_i) = y_i^3 \xi_i^2$, we have

$$\text{sp}(\ell'_i, \ell'_j) = y_j^3 \partial_j^2 \ell'_i - y_i^3 \partial_i^2 \ell'_j.$$

It is expressed as follows:

$$\begin{aligned} & \text{sp}(\ell'_i, \ell'_j) \\ &= (y_j^3 \partial_j^2 - \ell'_j) \ell'_i - (y_i^3 \partial_i^2 - \ell'_i) \ell'_j - (\ell'_i \ell'_j - \ell'_j \ell'_i) \\ &= (y_j^3 \partial_j^2 - \ell'_j) \ell'_i - (y_i^3 \partial_i^2 - \ell'_i) \ell'_j - (2\theta - a - b + 1)(\ell'_i - \ell'_j) \\ &= \{y_j^3 \partial_j^2 - \ell'_j - (2\theta - a - b + 1)\} \ell'_i \\ & \quad - \{y_i^3 \partial_i^2 - \ell'_i - (2\theta - a - b + 1)\} \ell'_j. \end{aligned} \quad (6)$$

Note that we have used the commutation relation

$$\ell'_i \ell'_j - \ell'_j \ell'_i = -(2\theta - a - b + 1)(\ell'_i - \ell'_j),$$

which is obtained by a straightforward calculation. By using the relations

$$\partial_i^2 \theta_i = 2\partial_i^2 + \theta_i \partial_i^2, \quad \partial_i^2 \theta_i^2 = 4(1 + y_i \partial_i) \partial_i^2 + \theta_i \partial_i^2,$$

we have

$$\begin{aligned} y_j^3 \partial_j^2 \ell'_i &= y_i^3 y_j^3 \partial_i^2 \partial_j^2 - y_j^3 \left\{ (-c_i + 2) y_i \partial_i + \sum_{k=1}^m \theta_k^2 + 4(1 + y_j \theta_j) \right. \\ & \quad \left. + \sum_{k \neq k'} \theta_k \theta_{k'} + 4 \sum_{k \neq j} \theta_k - (a + b) \left(\sum_{k=1}^m \theta_k + 2 \right) + ab \right\} \partial_j^2, \end{aligned}$$

which implies

$$\begin{aligned} & \text{in}_{\prec_w} \text{sp}(\ell'_i, \ell'_j) \\ &= \text{in}_{\prec_w} \left\{ -y_j^3 \left(\sum_{k, k'} y_k y_{k'} \partial_k \partial_{k'} \right) \partial_j^2 + y_i^3 \left(\sum_{k, k'} y_k y_{k'} \partial_k \partial_{k'} \right) \partial_i^2 \right\} \\ &= y_1^2 y_i^3 \xi_1^2 \xi_i^2 \end{aligned}$$

for $i < j$. On the other hand, since

$$\begin{aligned} & y_i^3 \partial_i^2 - \ell'_i - (2\theta - a - b + 1) \\ = & (c_i - 2)y_i \theta_i + (\theta - a)(\theta - b) - (2\theta - a - b + 1), \end{aligned}$$

we have

$$\begin{aligned} & \text{in}_{\prec_w} \{y_i^3 \partial_i^2 - \ell'_i - (2\theta - a - b + 1)\} \\ = & \text{in}_{\prec_w} \{(c_i - 2)y_i \theta_i + (\theta - a)(\theta - b) - (2\theta - a - b + 1)\} \\ = & \text{in}_{\prec_w} (\theta^2) \\ = & y_1^2 \xi_1^2. \end{aligned}$$

Note that it is independent of the index i . Hence we conclude that

$$\begin{aligned} \text{in}_{\prec_w} \{y_j^3 \partial_j^2 - \ell'_j - (2\theta - a - b + 1)\} \ell'_i &= y_1^2 y_i^3 \xi_1^2 \xi_i^2, \\ \text{in}_{\prec_w} \{y_i^3 \partial_i^2 - \ell'_i - (2\theta - a - b + 1)\} \ell'_j &= y_1^2 y_j^3 \xi_1^2 \xi_j^2, \end{aligned}$$

which imply that the expression (6) is a standard representation of $\text{sp}(\ell'_i, \ell'_j)$ with respect to the set $\{\ell'_1, \dots, \ell'_m\}$ and the order $\prec_{(\mathbf{0}, \mathbf{1})}$. It follows from the Buchberger's criterion that it is a Gröbner basis with respect to that order. Therefore the set

$$\{\text{in}_{(\mathbf{0}, \mathbf{1})}(\ell'_i) \mid i = 1, \dots, m\} = \left\{ y_i (y_i \xi_i)^2 - \left(\sum_{j=1}^m y_j \xi_j \right)^2 \mid i = 1, \dots, m \right\}$$

is a Gröbner basis of $\text{in}_{(\mathbf{0}, \mathbf{1})}(I'(m))$ by the theorem stated in [9, section 2] (the condition on the order can be weakened as in [12, Th 1.1.6]). In particular, it is a set of generators of the characteristic ideal $\text{in}_{(\mathbf{0}, \mathbf{1})}(I'(m))$. Q.E.D.

Let us determine the singular locus of ℓ'_1, \dots, ℓ'_m . The principal symbol L'_i of ℓ'_i is equal to

$$L'_i = y_i^3 \xi_i^2 - \left(\sum_{j=1}^m y_j \xi_j \right)^2.$$

When $y_i \neq 0$, it is factored as

$$L'_i = \left(y_i \sqrt{y_i} \xi_i - \sum_{j=1}^m y_j \xi_j \right) \left(y_i \sqrt{y_i} \xi_i + \sum_{j=1}^m y_j \xi_j \right).$$

We can show that the determinant of the coefficient matrix of the system

$$y_i \sqrt{y_i} \xi_i + \varepsilon_i \sum_{j=1}^m y_j \xi_j = 0, \quad i = 1, \dots, m$$

is equal to

$$\left(\prod_{j=1}^m y_j \sqrt{y_j} \right) \left(1 + \sum_{j=1}^m \frac{\varepsilon_j}{\sqrt{y_j}} \right) \quad (7)$$

by an analogous way to the proof of the Proposition 2.1. Therefore, the singular locus of $I'(m)$ is equal to the union of the zero sets of (7) where ε_j 's run over $\{-1, +1\}$. Thus, we have the following theorem.

Theorem 3.2 *The singular locus of $I(m)$ agrees with the zero set of (2) in the complex torus.*

4 Singular locus and the coordinate hyperplanes

We will prove that the coordinate hyperplanes are contained in the singular locus of $I(m)$ by discussing the cohomological solution sheaf $\mathcal{E}xt_{\mathcal{D}^{an}}^1(\mathcal{D}^{an}/\mathcal{D}^{an}I(m), \mathcal{O}^{an})$. We need a set of generators of the syzygies of $I(m)$ to describe the first cohomological solutions (as to an algorithmic method to determine it, see, e.g., [14]). We utilize a Gröbner basis with the order $\succ_{(-\mathbf{1}, \mathbf{1})}$, where $(-\mathbf{1}, \mathbf{1}) = (-1, \dots, -1, 1, \dots, 1)$ and the lexicographic order $\succ \partial_1 \succ \dots \succ \partial_m \succ x_1 \succ \dots \succ x_m$ as the tie-breaker, to determine the syzygies among generators of $I(m)$.

In order to use the S -pair criterion, we will work in the homogenized Weyl algebra $D^{(h)} = \mathbb{C}[h]\langle x_1, \dots, x_m, \partial_1, \dots, \partial_m \rangle$ (see, e.g., [12, §1.2]). The variable h is the homogenization variable which commutes with all other variables and we have the relation $\partial_i x_i = x_i \partial_i + h^2$.

Put

$$\begin{aligned} S_i &= \theta_i(\theta_i + (c_i - 1)h^2), \\ S_{ab} &= \left(\sum_{i=1}^m \theta_i + ah^2 \right) \left(\sum_{i=1}^m \theta_i + bh^2 \right) \end{aligned}$$

and

$$\begin{aligned} T_i &= hS_i - x_i S_{ab}, \\ T_{ij} &= x_j S_i - x_i S_j. \end{aligned}$$

They are homogeneous elements in the $D^{(h)}$. The operator T_i is the homogenization of ℓ_i and the operator T_{ij} is the homogenization of $x_j \ell_i - x_i \ell_j$. For two elements in $D^{(h)}$, their S -pair with respect to the order $\succ_{(-\mathbf{1}, \mathbf{1})}$ is defined similarly as in the section 3. We also use the terminology “standard representation” analogously for elements in $D^{(h)}$.

Theorem 4.1 *The set $G = \{T_1, \dots, T_m, T_{12}, T_{13}, \dots, T_{m-1, m}\}$ satisfies the S -pair criterion in the homogenized Weyl algebra $D^{(h)}$; G is a Gröbner basis with respect to the order $\succ_{(-\mathbf{1}, \mathbf{1})}$.*

Proof. We have the following standard representations of S -pairs in terms of G :

$$\text{sp}(T_i, T_j) = S_j T_i - S_i T_j = S_{a-1, b-1} T_{ij}, \quad (8)$$

$$\text{sp}(T_i, T_{ij}) = x_j T_i - h T_{ij} = x_i T_j, \quad (9)$$

$$\begin{aligned} \text{sp}(T_j, T_{ij}) &= x_i^2 \partial_i^2 T_j - h x_j \partial_j^2 T_{ij} \\ &= \{x_i(x_j^{-1} S_j) - c_i h^2 \theta_i\} T_j - (2h^2 \theta_j + c_j h^4) T_i, \end{aligned} \quad (10)$$

$$\begin{aligned} \text{sp}(T_k, T_{ij}) &= x_i^2 x_j \partial_i^2 T_k - h x_k^2 \partial_k^2 T_{ij} \\ &= h S_j T_{ki} + x_k S_i T_j - c_i h^2 x_j \theta_i T_k + c_k h^3 \theta_k T_{ij}, \end{aligned} \quad (11)$$

$$\text{sp}(T_{ij}, T_{ik}) = x_k T_{ij} - x_j T_{ik} = -x_i T_{jk}, \quad (12)$$

$$\text{sp}(T_{ij}, T_{kj}) = x_k^2 \partial_k^2 T_{ij} - x_i^2 \partial_i^2 T_{kj} = S_j T_{ik} - c_k h^2 \theta_k T_{ij} + c_i h^2 \theta_i T_{kj}, \quad (13)$$

$$\begin{aligned} \text{sp}(T_{ij}, T_{jk}) &= x_j x_k \partial_j^2 T_{ij} - x_i^2 \partial_i^2 T_{jk} \\ &= \{S_k + (2 - c_j) h^2 x_k \partial_j\} T_{ij} + (c_j - 2) h^4 T_{ik} \\ &\quad + \{(2 - c_j) h^2 x_i \partial_j + c_i h^2 \theta_i - x_i \theta_j \partial_j\} T_{jk}, \end{aligned} \quad (14)$$

$$\begin{aligned} \text{sp}(T_{ij}, T_{i'j'}) &= x_{i'}^2 x_{j'} \partial_{i'}^2 T_{ij} - x_i^2 x_j \partial_i^2 T_{i'j'} \\ &= x_{j'} S_j T_{ii'} - x_{i'} S_i T_{jj'} - c_{i'} h^2 x_{j'} \theta_{i'} T_{ij} + c_i h^2 x_j \theta_i T_{i'j'}, \end{aligned} \quad (15)$$

where we assume that the indices i, j, k, i', j' satisfy $i \neq k, j \neq k$ and $\{i, j\} \cap \{i', j'\} = \emptyset$. Note in the above that we regard $T_{ji} = -T_{ij}$ for $i < j$. Thus, we have proved the set G is a Gröbner basis. Q.E.D.

By [10, Th 9.10], syzygies are generated by the dehomogenizations of the standard representations of the S -pairs. The following Corollary will be used to complete the proof of our main theorem.

Corollary 4.2 *The set of relations which are derived from the standard representations of the s -pairs gives a set of generators of the syzygies among ℓ_i , ($i = 1, \dots, m$), $\ell_{ij} = x_j \ell_i - x_i \ell_j$, $1 \leq i < j \leq m$. For example, (9) yields the syzygy $x_j \ell_i - \ell_{ij} - x_i \ell_j = 0$.*

Theorem 4.3 *The singular locus of $I(m)$ is the zero set of (2).*

Proof. It follows from the discussions in the section 2 that we may prove only that $x_i = 0$ are contained in the singular locus. Let us consider if $x_m = 0$ is a singular locus or not. Let $g_m(x')$ be a non-zero solution of $I(m-1)$. This function does not depend on x_m . Put $g_1 = \dots = g_{m-1} = 0$. Then, we have $\ell_i \cdot g_m = 0$ for $i \neq m$ and $\ell_j \cdot g_i = 0$ for $i = 1, \dots, m-1$. Define $g_{ij} = x_j g_i - x_i g_j$. $\sum g_i e_i + \sum g_{ij} e_{ij}$ are annihilated by the generators of the syzygies given in the Corollary 4.2. For instance, we have the syzygy

$$(\theta_j(\theta_j - 1) + c_j \theta_j) \ell_i - (\theta_i(\theta_i - 1) + c_i \theta_i) \ell_j - (\theta + a - 1)(\theta + b - 1) \ell_{ij} = 0$$

by the equation (8). For $i < j = m$, we have

$$\begin{aligned}
& (\theta_j(\theta_j - 1) + c_j\theta_j)g_i - (\theta_i(\theta_i - 1) + c_i\theta_i)g_j - (\theta + a - 1)(\theta + b - 1)g_{ij} \\
= & -(\theta_i(\theta_i - 1) + c_i\theta_i)g_m - (\theta + a - 1)(\theta + b - 1)(-x_i g_m) \\
= & -(\theta_i(\theta_i - 1) + c_i\theta_i)g_m + x_m(\theta + a)(\theta + b)g_m \\
= & -\left\{(\theta_i(\theta_i - 1) + c_i\theta_i) - x_m\left(\sum_{k=1}^{m-1}\theta_k + a\right)\left(\sum_{k=1}^{m-1}\theta_k + b\right)\right\}g_m \\
& \quad (\text{since } g_m = g_m(x') \text{ does not depend on } x_m, \text{ we have } \partial_m g_m = 0) \\
= & 0.
\end{aligned}$$

When $i < j < m$, obviously the equation

$$(\theta_j(\theta_j - 1) + c_j\theta_j)g_i - (\theta_i(\theta_i - 1) + c_i\theta_i)g_j - (\theta + a - 1)(\theta + b - 1)g_{ij} = 0$$

holds.

Let us try to solve $\ell_i \cdot f = g_i$, $i = 1, \dots, m$ and $\ell_{ij} \cdot f = g_{ij}$, $1 \leq i < j \leq m$. The second group of equation is solved when the first group is solved. Put $f = \sum_{k=0}^{\infty} f_k x_m^k$. The left hand side of $\ell_m f$ can be factored by x_m . On the other hand, the right hand side g_m is nonzero and does not depend on x_m . Therefore the system $\ell_i \cdot f = g_i$ does not have a holomorphic solution along $x_m = 0$. Therefore, we have proved that $\mathcal{E}xt_{\mathcal{D}}^1(\mathcal{D}^{an}/\mathcal{D}^{an}E_C, \mathcal{O})$ is not zero at a generic point in $x_m = 0$. By Kashiwara's theorem [8, Theorem 4.1], the $\mathcal{E}xt^1$ must be zero if $x_m = 0$ is not a singular locus. Thus, we have proved that $x_m = 0$ is the singular locus. We can analogously show that other varieties $x_i = 0$ are also contained in the singular locus. Q.E.D.

5 The A -hypergeometric system associated to the Lauricella F_C

The binomial D -modules [3] are introduced to study classical hypergeometric systems including the Lauricella F_C . The contents of the first half part of this section are implicitly or explicitly explained in [3], but they do not seem to be publicized to people who study classical Lauricella functions and related topics. We add the first part of this section to explain how to apply the theory of A -hypergeometric systems and binomial D -modules to study F_C . The second part contains a new result and utilizes the first part; The last Theorem 5.3 describes the singular locus for the A -hypergeometric system associated to F_C in the complex torus. The singular locus is the zero set of the principal A -determinant [5] for the A associated to F_C .

We denote $I(m)$, which annihilates the Lauricella function F_C , by E_C in this section. For a given Heun system, there exists a corresponding binomial D -module. In case of E_C , the corresponding binomial system is an A -hypergeometric system. Let us study this system.

Let $e_1, \dots, e_{m+1}, e_{m+2}$ be the standard basis of \mathbb{Z}^{m+2} . Following [11], consider the set of points $\mathcal{A} = \{e_1 + e_{m+2}, e_2 + e_{m+2}, \dots, e_{m+1} + e_{m+2}, -e_1 + e_{m+2}, -e_2 + e_{m+2}, \dots, -e_{m+1} + e_{m+2}\}$. We define a matrix $A(F_C, m)$ consisting of these points as column vectors. The matrix is an $(m+2) \times 2(m+1)$ matrix. For example, we have

$$A(F_C, 2) = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Let $H_A(\beta)$ be the A -hypergeometric system associated to the matrix $A(F_C, m)$, the parameter $\beta^T = (1 - c_1, \dots, 1 - c_m, b - a, \sum_{j=1}^m c_j - a - b - m)$ and the variables $u_1, \dots, u_{m+1}, u_{-1}, \dots, u_{-(m+1)}$ as independent variables. The associated differential operators for u_j and u_{-j} are denoted by ∂_j and ∂_{-j} respectively. The toric ideal I_A defined by $A = A(F_C, m)$, $I_A = \{\partial^u - \partial^v \mid Au = Av, u, v \in \mathbb{N}_0^{2m+2}\}$, is generated by $\partial_j \partial_{-j} - \partial_{m+1} \partial_{-(m+1)}$, $j = 1, \dots, m$ in $\mathbb{C}[\partial_1, \dots, \partial_{m+1}, \partial_{-1}, \dots, \partial_{-(m+1)}]$.

The left ideal $H_A(\beta)$ is generated by the row vectors of $A\theta_u - \beta$ and I_A where $\theta_u = (u_1 \partial_1, \dots, u_{m+1} \partial_{m+1}, u_{-1} \partial_{-1}, \dots, u_{-(m+1)} \partial_{-(m+1)})^T$. We denote the i -th row vector of $A\theta - \beta$ by $E_i - \beta_i$.

Theorem 5.1 ([3]) *Sol(E_C) \simeq Sol($H_A(\beta)$) holds for any parameter in a simply connected neighborhood of any generic point. In particular, the holonomic rank of $E_C = I(m)$ is equal to 2^m .*

Proof. Let F be a solution of E_C . Following [11], we consider the following function

$$f(u) = u_{m+1}^{-a} u_{-(m+1)}^{-b} \prod_{j=1}^m u_{-j}^{c_j-1} F\left(\frac{u_1 u_{-1}}{u_{m+1} u_{-(m+1)}}, \dots, \frac{u_m u_{-m}}{u_{m+1} u_{-(m+1)}}\right) \quad (16)$$

Let us prove that the function $f(u)$ is a solution of $H_A(\beta)$. It is easy to see that $(E_i - \beta_i) \cdot f = 0$. Put $\eta = u_{m+1}^{-a} u_{-(m+1)}^{-b} \prod_{j=1}^m u_{-j}^{c_j-1}$ and $z_j = \frac{u_j u_{-j}}{u_{m+1} u_{-(m+1)}}$. We have

$$\begin{aligned} \theta_j \theta_{-j} \cdot f(u) &= \theta_j \cdot ((c_j - 1)\eta F + \eta z_j F_j) \\ &= (c_j - 1)\eta z_j F_j + \eta z_j F_j + \eta z_j z_j F_{jj} \\ &= \eta \theta_{z_j} (\theta_{z_j} + c_j - 1) \cdot F(z) \end{aligned} \quad (17)$$

where F_j denotes the partial derivative of $F(z_1, \dots, z_m)$ with respect to the variable z_j . Analogously, we get

$$\theta_{m+1} \theta_{-(m+1)} \cdot f(u) = \eta \left(\sum_{i=1}^m \theta_{z_i} + a \right) \left(\sum_{i=1}^m \theta_{z_i} + b \right) \cdot F(z) \quad (18)$$

Put $\tilde{\ell}_j = u_j u_{-j} u_{m+1} u_{-(m+1)} (\partial_j \partial_{-j} - \partial_{m+1} \partial_{-(m+1)})$. This is equal to

$$u_{m+1} u_{-(m+1)} \theta_j \theta_{-j} - u_j u_{-j} \theta_{m+1} \theta_{-(m+1)}.$$

It follows from (17) and (18) that we have $\tilde{\ell}_j \cdot f(u) = 0$, which implies that $f(u)$ is a solution of $H_A(\beta)$. Note that the correspondence from F to f is an injection among \mathbb{C} -vector spaces on a simply connected open set.

Conversely, let f be a solution of $H_A(\beta)$. We define new $2m+2$ variables z_j by

$$\begin{aligned} z_j &= u_j u_{-j} / (u_{m+1} u_{-(m+1)}), \quad z_{m+j} = u_{-j}^{-1}, \quad j = 1, \dots, m, \\ z_{2m+1} &= u_{m+1}^{-1}, \quad z_{2m+2} = u_{-(m+1)}^{-1} \end{aligned} \quad (19)$$

Note that this gives an isomorphism of the complex toruses $(\mathbb{C}^*)^{2m+1} = \{u\}$ and $(\mathbb{C}^*)^{2m+1} = \{z\}$. The Euler operator $\theta_{\pm u_j} = u_{\pm j} \partial_{\pm j}$ can be written as a sum of Euler operators with respect to z_i 's. In fact, we have

$$\begin{aligned} u_j \partial_j &= z_j \partial_{z_j}, \quad u_{-j} \partial_{-j} = z_j \partial_{z_j} - z_{m+j} \partial_{z_{m+j}} \\ u_{m+1} \partial_{m+1} &= - \sum_{k=1}^m z_j \partial_{z_k} - z_{2m+1} \partial_{z_{2m+1}} \\ u_{-(m+1)} \partial_{-(m+1)} &= - \sum_{k=1}^m z_j \partial_{z_k} - z_{2m+2} \partial_{z_{2m+2}}. \end{aligned}$$

Put $f' = \eta^{-1} f$. The equations $(E_i - \beta_i) \cdot f = 0$ yield $\theta_{z_j} \cdot f' = 0$ for $j = m+1, \dots, 2(m+1)$. This implies that f' depends only on z_1, \dots, z_m . An analogous calculation with (17) and (18) yields the equation $\ell_i \cdot f'(u(z)) = 0$. This means that the map $F(z) \mapsto f(u)$ is surjective. Thus, we have proved $\text{Sol}(E_C) \simeq \text{Sol}(H_A(\beta))$.

The correspondence gives the holonomic rank of E_C by evaluating the degree of I_A [4]. Since $\{\partial_j \partial_{-j} - \partial_{m+1} \partial_{-(m+1)} \mid j = 1, \dots, m\}$ is a Gröbner basis of I_A , the degree is equal to that of the monomial ideal generated by $\partial_j \partial_{-j}$, $j = 1, \dots, m$. This degree is equal to 2^m . Q.E.D.

An interesting application of this isomorphism is the following irreducibility condition of E_C . We can utilize the recent result by Beukers [1] and Schulze and Walther [13] on irreducibility of A -hypergeometric systems to give a condition of the irreducibility of E_C .

Theorem 5.2 ([1], [13]) *The system E_C is irreducible if and only if*

$$\frac{1}{2} \left(\sum_{i=1}^m c_i - a - b - 2 \sum_{i=1}^m \varepsilon_i (1 - c_i) + \varepsilon_{m+1} (b - a) \right) \notin \mathbb{Z}$$

for all combinations of $\varepsilon_i \in \{-1, 1\}$.

Proof. It follows from the previous theorem that the irreducibility of E_C is equivalent to that of $H_A(\beta)$. In fact, the solution space is locally isomorphic and differential operators with rational function coefficients in z is mapped to those in u by (19). The primitive integral support functions $P_J(s)$ for $A(F_C, m)$ are $(1/2)(s_{m+2} + \sum_{j \in J} s_j - \sum_{j \notin J} s_j)$, $J \subseteq [1, m+1]$ where s_j 's are the dual basis for the e_i 's [11]. It follows from [1] or [13] that the irreducibility condition is that $P_J(\beta) \notin \mathbb{Z}$ for all J , which is equivalent to the condition in the theorem. Q.E.D.

Finally, we discuss the singular locus of the $H_A(\beta)$ via the correspondence. The correspondence is not only for the classical solutions as we have seen, but also for some D -module invariants including the singular locus on the complex torus. In this case, we utilize our result on the singular locus for F_C to derive a result on the A -hypergeometric system.

Theorem 5.3 *The singular locus of $H_A(\beta)$ in the complex torus is given by the zero set of*

$$\prod_{\varepsilon_i \in \{-1, 1\}} \left(1 + \varepsilon_1 \sqrt{\frac{u_1 u_{-1}}{u_{m+1} u_{-(m+1)}}} + \cdots + \varepsilon_m \sqrt{\frac{u_m u_{-m}}{u_{m+1} u_{-(m+1)}}} \right)$$

Proof. We denote by D_{2m+2}^* the ring of differential operators on the complex torus

$$\mathbb{C}\langle z_1^\pm, \dots, z_{2m+2}^\pm, \partial_{z_1}, \dots, \partial_{z_{2m+2}} \rangle.$$

Let I be a left ideal in D_{2m+2}^* and for a complex number α denote by $D_{2m+2}^* z_{2m+2}^\alpha$ the left $\mathbb{C}\langle z_{2m+2}^\pm, \partial_{z_{2m+2}} \rangle$ -module $\mathbb{C}\langle z_{2m+2}^\pm, \partial_{z_{2m+2}} \rangle / \langle z_{2m+2} \partial_{z_{2m+2}} - \alpha \rangle$. The outer tensor product $(D_{2m+2}^*/I) \boxtimes D_{2m+2}^* z_{2m+2}^\alpha$ is defined by the restriction of

$$D_{2m+3}^* / \langle I, z_{2m+3} \partial_{z_{2m+3}} - \alpha \rangle$$

to $z_{2m+3} - z_{2m+2} = 0$. In other words,

$$\begin{aligned} & (D_{2m+2}^*/I) \boxtimes D_{2m+2}^* z_{2m+2}^\alpha |_{z_{2m+2} \mapsto t, \partial_{z_{2m+2}} \mapsto \partial_t} \\ & \simeq D_{2m+1}^* \langle t^\pm, s, \partial_t, \partial_s \rangle / (\langle I, -t \partial_s + s \partial_s - \alpha \rangle + s D_{2m+1}^* \langle t^\pm, s, \partial_t, \partial_s \rangle) \end{aligned} \quad (20)$$

where we make replacements

$$s = z_{2m+2} - z_{2m+3}, t = z_{2m+2}, z_{2m+2} \partial_{z_{2m+2}} = t \partial_s + t \partial_t, z_{2m+3} \partial_{z_{2m+3}} = -t \partial_s + s \partial_s$$

in I . Let $u = (0, \dots, 0, 1)$ be the weight vector where 1 stands for the variable s . Then, $b = \text{in}_{(-u, u)}(-t \partial_s + s \partial_s - \alpha) = -t \partial_s$. Since t is invertible, we have $D_{2m+1}^* \langle t^\pm, s, \partial_t, \partial_s \rangle b \cap \mathbb{C}[s \partial_s] = \langle s \partial_s \rangle$. Therefore, by the restriction algorithm (see, e.g., [10]), we can prove that (20) is isomorphic to

$$D_{2m+1}^* \langle t^\pm, \partial_t \rangle / ((\langle I, -t \partial_s + s \partial_s - \alpha \rangle + s D_{2m+1}^* \langle t^\pm, s, \partial_t, \partial_s \rangle) \cap D_{2m+1}^* \langle t^\pm, \partial_t \rangle).$$

of which denominator is called the restriction ideal.

Put $\tau_j = z_j \partial_{z_j}$. Let I be the hypergeometric ideal $H_A(\beta)$ expressed in terms of the variable in z_j (19), which is generated in D_{2m+2}^* by

$$\tau_j(\tau_j - \tau_{m+j}) - z_j \left(\sum_{j=1}^m \tau_j + \tau_{2m+1} \right) \left(\sum_{j=1}^m \tau_j + \tau_{2m+2} \right), \quad j = 1, \dots, m$$

and

$$\tau_{m+j} - (1 - c_j), \quad j = 1, \dots, m, \quad \tau_{2m+2} - \tau_{2m+1} - (b - a), \tau_{2m+1} - a.$$

Note that $\tau_j(\tau_j - \tau_{m+j}) - z_j(\sum_{j=1}^m \tau_j + \tau_{2m+1})(\sum_{j=1}^m \tau_j + t\partial_s + t\partial_t)$ is in I under the change of variables from z_{2m+2}, z_{2m+3} to s, t . Subtracting $(\sum \tau_j + \tau_{2m+1})(-t\partial_s + s\partial_s - \alpha)$ from it, we conclude that the restriction ideal contains

$$\tau_j(\tau_j - \tau_{m+j}) - z_j \left(\sum \tau_j + \tau_{2m+1} \right) \left(\sum \tau_j + t\partial_t - \alpha \right). \quad (21)$$

We have defined the outer tensor product by $D^* z_{2m+2}^\alpha$ and studied its properties. We can make analogous discussions for outer tensor products by other variables and we conclude from (21) that there exists a left ideal I' such that

$$(D_{2m+2}^*/I) \boxed{\times} D^* \eta^{-1} \simeq D_{2m+2}^*/I' \quad (22)$$

and $I' \supseteq \langle I(m), \partial_{z_{m+1}}, \dots, \partial_{z_{2m+2}} \rangle$. Here, $I(m)$ is the left ideal generated by the Lauricella operators (1) (x_i 's are replaced by z_i 's respectively).

We denote by $\text{Sing}^*(M)$ the singular locus of M in the complex torus. It follows from (22) that

$$\begin{aligned} \text{Sing}^*(D_{2m+2}^*/I) &= \text{Sing}^*((D_{2m+2}^*/I) \boxed{\times} D^* \eta^{-1}) \\ &= \text{Sing}^*(D_{2m+2}^*/I') \subseteq \text{Sing}^*(D^*/I(m)) \end{aligned} \quad (23)$$

Since $\text{Sing}^*(D^*/I(m))$ is irreducible, the singular locus of the A -hypergeometric system $\text{Sing}^*(D_{2m+2}^*/I)$ is empty or agrees with $\text{Sing}^*(D^*/I(m))$. Since the toric ideal I_A is Cohen-Macaulay when $A = A(F_C, m)$, the singular locus of the A -hypergeometric system $H_A(\beta)$ does not depend on the parameter β [4], [12, Section 4.3]. Then, we may suppose that $H_A(\beta)$ is irreducible. By [6], the A -hypergeometric system is regular holonomic and the irreducibility implies that the irreducibility of the monodromy representation. If the singular locus in the complex torus is empty, then the monodromy representation is reducible. Then, the singular locus is not empty and then we obtain the conclusion. Q.E.D.

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